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A THERMOELASTIC PROBLEM FOR A CRACK BETWEEN DISSIMILAR ANISOTROPIC MEDIA

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Abstract—The thermoelastic problem of a flat crack of infinite length and constant finite width between bonded dissimilar anisotropic materials is examined. The stress singularities at the crack tips are obtained and compared with those for the corresponding elastic crack problem.

1. INTRODUCTION

Over recent years a number of generalised plane thermoelastic boundary value problems have been solved for anisotropic materials. Thus, e.g. Clements[1], Tauchert and Aköz[2], Clements and Toy[3], Atkinson and Clements[4] and Clements and Tauchert[5] have successfully employed integral transform techniques in order to solve various thermoelastic contact, crack and slab problems for anisotropic media. So far complex variable techniques have apparently not been used for the solution of thermoelastic problems involving general anisotropy. Since in many cases the problems which can be solved using integral transforms can also be solved by complex variable methods it seems of little importance which of the two procedures is used. However there are certain problems which only readily yield to one or other of the two techniques. For example the slab problems considered in Tauchert and Aköz[2] and Clements and Tauchert[5] are much more easily solved by employing integral transforms while problems involving cracks between dissimilar media are extremely difficult to solve by using transform techniques. In fact the thermoelastic problem associated with a crack between dissimilar media has not so far been solved. Thus the purpose of the present paper is to first set up the necessary expressions for the temperature, heat flux, displacement and stress in terms of arbitrary analytic functions and then to use these representations to solve for the flux and stress fields round a crack between dissimilar anisotropic media. This problem is of some importance since layered anisotropic composites may contain crack like flows along a bonded interface and the present analysis could be used to examine the nature of the thermally induced stress field in the vicinity of such a crack.

2. STATEMENT OF THE PROBLEM

Taking Cartesian coordinates x_1, x_2, x_3 and assume the two dissimilar anisotropic materials occupy the regions $x_2>0$ and $x_2<0$ which will be denoted by L and R respectively. The materials are assumed to be bonded at all points of the interface $x_2=0$ except those lying in the region $|x_1|\leq a$, $-\infty<x_3<\infty$ where there is a crack. On the two faces of the crack equal and opposite heat fluxes and tractions are specified. It is required to find the temperature, flux, displacement and stress fields in the bonded material.

If the temperature, heat flux, displacement and stress in the regions L and R are denoted by T^L, $-P^L$, u_k^L, σ_{ij}^L and T^R, $-P^R$, u_k^R, σ_{ij}^R respectively then the following conditions must be satisfied on $x_2=0$:

$$P^L=-f(x_1) \text{ on } x_2=0+ \text{ for } |x_1|<a, \tag{2.1}$$

$$P^R=-f(x_1) \text{ on } x_2=0- \text{ for } |x_1|<a, \tag{2.2}$$

$$\sigma_{i2}^L=-p_i(x_1) \text{ on } x_2=0+ \text{ for } |x_1|<a, \tag{2.3}$$

$$\sigma_{i2}^R=-p_i(x_1) \text{ on } x_2=0- \text{ for } |x_1|<a, \tag{2.4}$$

and

$$u_k^L = u_k^R \text{ on } x_2 = 0 \text{ for } |x_1| > a, \tag{2.5}$$

$$T^L = T^R \text{ on } x_2 = 0 \text{ for } |x_1| > a, \tag{2.6}$$

$$P^L = P^R \text{ on } x_2 = 0 \text{ for } |x_1| > a, \tag{2.7}$$

$$\sigma_{i2}^L = \sigma_{i2}^R \text{ on } x_2 = 0 \text{ for } |x_1| > a. \tag{2.8}$$

where $f(x_1)$ is the given heat flux and $p_i(x_1)$ the given tractions over the crack faces.

3. FUNDAMENTAL EQUATIONS

Consider a homogeneous anisotropic elastic solid in which the displacement, stress and temperature fields are independent of the Cartesian coordinate x_3. The temperature distribution $T(x_1, x_2)$ in the material satisfies the heat conduction equation

$$\lambda_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} = 0, \tag{3.1}$$

where $\lambda_{ij} = \lambda_{ji}$ are the coefficients of heat conduction and the repeated suffix summation convention (summing from 1 to 3 for Latin suffices only) has been used. The general solution to (3.1) in terms of an arbitrary analytic function χ is (see Clements[1])

$$T(x_1, x_2) = \chi(z') + \bar{\chi}(\bar{z}') \tag{3.2}$$

where the bar denotes the complex conjugate and $z' = x_1 + \tau x_2$ where τ is the root with positive imaginary part of the quadratic equation

$$\lambda_{11} + 2\lambda_{12}\tau + \lambda_{22}\tau^2 = 0. \tag{3.3}$$

The stress σ_{ij} induced in the material by the temperature distribution (3.2) is related to the elastic displacements u_k by the equations

$$\sigma_{ij} = c_{ijkl} \frac{\partial u_k}{\partial x_l} - \beta_{ij} T \tag{3.4}$$

where c_{ijkl} are the elastic constants and β_{ij} are the stress-temperature coefficients. The stresses σ_{ij} given by (3.4) must satisfy the equilibrium equations $\partial \sigma_{ij} / \partial x_j$ and hence

$$c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} - \beta_{ij} \frac{\partial T}{\partial x_j} = 0. \tag{3.5}$$

Since T is given by (3.2) we try for a solution to (3.5) in the form

$$u_k = C_k \phi(z') + \bar{C}_k \bar{\phi}(\bar{z}'), \tag{3.6}$$

where the C_k are constants and

$$\phi'(z) = \chi(z). \tag{3.7}$$

The displacement (3.6) will be a solution to (3.5) if

$$D_{ik} C_k = \gamma_i \tag{3.8}$$

where

$$D_{ik} = c_{i1k1} + \tau c_{i1k2} + \tau c_{i2k1} + \tau^2 c_{i2k2} \tag{3.9}$$

and

$$\gamma_i = \beta_{i1} + \beta_{i2}. \tag{3.10}$$

Equations (3.8) serves to determine the constants C_k.

In addition to the displacement which is given by (3.6) we may add any displacement which is a solution to the equation

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0. \tag{3.11}$$

Solutions to this homogeneous system may be written in the form (see Clements[6])

$$u_k = \sum_\alpha A_{k\alpha}\psi_\alpha(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha}\bar{\psi}_\alpha(\bar{z}_\alpha) \tag{3.12}$$

where the sum is from 1 to 3, the $\psi_\alpha(z)$ are arbitrary analytic functions and $z_\alpha = x_1 + p_\alpha x_2$ where p_1, p_2, p_3 are the roots with positive imaginary part of the sextic

$$|c_{i1k1} + pc_{i1k2} + pc_{i2k1} + p^2 c_{i2k2}| = 0. \tag{3.13}$$

Also the $A_{k\alpha}$ are the solutions of the equations

$$(c_{i1k1} + p_\alpha c_{i1k2} + p_\alpha c_{i2k1} + p_\alpha{}^2 c_{i2k2})A_{k\alpha} = 0. \tag{3.14}$$

From (3.6) and (3.12) the displacement may now be written in the form

$$u_k = \sum_\alpha A_{k\alpha}\psi_\alpha(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha}\bar{\psi}_\alpha(\bar{z}_\alpha) + C_k\phi(z') + \bar{C}_k\bar{\phi}(\bar{z}') \tag{3.15}$$

Hence, from (3.4), (3.15), (3.2) and (3.7) the stress may be written in the form

$$\sigma_{ij} = \sum_\alpha L_{ij\alpha}\psi_\alpha'(z_\alpha) + \sum_\alpha \bar{L}_{ij\alpha}\bar{\psi}_\alpha'(\bar{z}_\alpha) + (N_{ij} - \beta_{ij})\phi'(z') + (\bar{N}_{ij} - \beta_{ij})\bar{\phi}'(\bar{z}') \tag{3.16}$$

where the primes on the analytic functions indicate differentiation with respect to the argument in question and

$$L_{ij\alpha} = (c_{ijk1} + p_\alpha c_{ijk2})A_{k\alpha} \tag{3.17}$$

$$N_{ij} = (c_{ijk1} + \tau c_{ijk2})C_k. \tag{3.18}$$

Various properties of the constants occurring in (3.15) and (3.16) may be determined by employing procedures outlined in Clements[1].

It is possible to present the analysis in a more compact form if the expressions for the temperature, displacement and stress are cast into an alternative form. Suppose the upper half-space L is occupied by a material with constants λ_{ij}^L, c_{ijkl}^L, $A_{k\alpha}^L$, $L_{ij\alpha}^L$, $C_k{}^L$, N_{ij}^L, β_{ij}^L and the lower half-space R by a material with constants for which the superscript R is attached. Define

$$(\lambda_{21}^L + \tau^L\lambda_{22}^L)\phi(z) = \Phi(z) \text{ for } z \in L, \tag{3.19}$$

$$(\lambda_{21}^R + \tau^R\lambda_{22}^R)\phi(z) = \Psi(z) \text{ for } z \in R. \tag{3.20}$$

Hence

$$\phi(z) = E^L\Phi(z) \text{ for } z \in L, \tag{3.21}$$

$$\phi(z) = E^R\Psi(z) \text{ for } z \in R, \tag{3.22}$$

where

$$E^L = (\lambda_{21}^L + \tau^L \lambda_{22}^L)^{-1}, \qquad E^R = (\lambda_{21}^R + \tau^R \lambda_{22}^R)^{-1}. \tag{3.23}$$

Then, from (3.2) and (3.7)

$$T = E^L \Phi'(z') + \bar{E}^L \bar{\Phi}'(\bar{z}') \text{ for } z' \in L, \tag{3.24}$$

$$T = E^R \Psi'(z') + \bar{E}^R \bar{\Psi}'(\bar{z}') \text{ for } z' \in R. \tag{3.25}$$

Also define

$$\sum_\alpha L_{i2\alpha}^L \psi_\alpha(z) = \Omega_i(z) \text{ for } z \in L \tag{3.26}$$

$$\sum_\alpha L_{i2\alpha}^R \psi_\alpha(z) = \Theta_i(z) \text{ for } z \in R, \tag{3.27}$$

where the $\Omega_i(z)$ and $\Theta_i(z)$ are defined and analytic in L and R respectively. Stroh[7] has shown that the matrix $[L_{i2\alpha}]$ is non-singular and hence

$$\psi_\alpha(z) = M_{\alpha j}^L \Omega_j(z) \text{ for } z \in L, \tag{3.28}$$

$$\psi_\alpha(z) = M_{\alpha j}^R \Theta_j(z) \text{ for } z \in R, \tag{3.29}$$

where

$$\sum_\alpha L_{i2\alpha}^L M_{\alpha j}^L = \delta_{ij}, \qquad \sum_\alpha L_{i2\alpha}^R M_{\alpha j}^R = \delta_{ij}. \tag{3.30}$$

Hence, in (3.15) and (3.16)

$$u_k^L = \sum_\alpha A_{k\alpha}^L M_{\alpha j}^L \Omega_j(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha}^L \bar{M}_{\alpha j}^L \bar{\Omega}_j(\bar{z}_\alpha) + C_k^L E^L \Phi(z') + \bar{C}_k^L \bar{E}^L \bar{\Phi}(\bar{z}') \text{ for } z_\alpha, z' \in L, \tag{3.31}$$

$$\begin{aligned}\sigma_{ij}^L = &\sum_\alpha L_{ij\alpha}^L M_{\alpha k}^L \Omega_k'(z_\alpha) + \sum_\alpha \bar{L}_{ij\alpha}^L \bar{M}_{\alpha k}^L \bar{\Omega}_k'(\bar{z}_\alpha) + (N_{ij}^L - \beta_{ij}^L) E^L \Phi'(z') \\ &+ (\bar{N}_{ij}^L - \beta_{ij}^L) \bar{E}^L \bar{\Phi}'(\bar{z}') \text{ for } z_\alpha, z' \in L,\end{aligned} \tag{3.32}$$

$$\begin{aligned}u_k^R = &\sum_\alpha A_{k\alpha}^R M_{\alpha j}^R \Theta_j(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha}^R \bar{M}_{\alpha j}^R \bar{\Theta}_j(\bar{z}_\alpha) + C_k^R E^R \Psi(z') \\ &+ \bar{C}_k^R \bar{E}^R \bar{\Psi}(\bar{z}') \text{ for } z_\alpha, z' \in R,\end{aligned} \tag{3.33}$$

$$\begin{aligned}\sigma_{ij}^R = &\sum_\alpha L_{ij\alpha}^R M_{\alpha k}^R \Theta_k'(z_\alpha) + \sum_\alpha \bar{L}_{ij\alpha}^R \bar{M}_{\alpha k}^R \bar{\Theta}_k'(\bar{z}_\alpha) + (N_{ij}^R - \beta_{ij}^R) E^R \Psi'(z') \\ &+ (\bar{N}_{ij}^R - \beta_{ij}^R) \bar{E}^R \bar{\Psi}'(\bar{z}') \text{ for } z_\alpha, z' \in R.\end{aligned} \tag{3.34}$$

4. TEMPERATURE FIELD

The heat flux $-P$ at a point across the surface with outward normal $\mathbf{n} = (n_1, n_2, 0)$ is given by

$$P = \lambda_{1j} \frac{\partial T}{\partial x_j} n_1 + \lambda_{2j} \frac{\partial T}{\partial x_j} n_2. \tag{4.1}$$

Hence, from (3.23) to (3.25) it follows that the flux will be continuous across the plane $x_2 = 0$ outside the cut from $x_1 = -a$ to $x_1 = a$ if

$$\Phi''^{+}(x_1) + \bar{\Phi}''^{-}(x_1) = \Psi''^{-}(x_1) + \bar{\Psi}''^{+}(x_1) \text{ for } |x_1| > a, \tag{4.2}$$

or

$$\Phi''^{+}(x_1) - \bar{\Psi}''^{+}(x_1) = \Psi''^{-}(x_1) - \bar{\Phi}''^{-}(x_1) \text{ for } |x_1| > a. \tag{4.3}$$

where

$$\lim_{x_2 \to 0+} \Phi(z) = \Phi^{+}(x_1), \qquad \lim_{x_2 \to 0-} \Phi(z) = \Phi^{-}(x_1).$$

Thus, if we put

$$\Phi''(z) - \bar{\Psi}''(z) = \Lambda(z) \text{ for } z \in L, \tag{4.4}$$

$$\Psi''(z) - \bar{\Phi}''(z) = \Lambda(z) \text{ for } z \in R, \tag{4.5}$$

where $\Lambda(z)$ is analytic in the whole plane cut along $(-a, a)$ then eqn (4.3) is identically satisfied. Similarly, the temperature will be continuous across the bonded interface if

$$E^{L}\Phi'(z) - \bar{E}^{R}\bar{\Psi}'(z) = \text{H}(z) \text{ for } z \in L, \tag{4.6}$$

$$E^{R}\Psi'(z) - \bar{E}^{L}\bar{\Phi}'(z) = \text{H}(z) \text{ for } z \in R. \tag{4.7}$$

Differentiating (4.6) and (4.7) and substituting from (4.4) and (4.5)

$$(E^{L} - \bar{E}^{R})\Phi''(z) = \text{H}'(z) - \bar{E}^{R}\Lambda(z) \text{ for } z \in L, \tag{4.8}$$

$$(E^{R} - \bar{E}^{L})\bar{\Phi}''(z) = \text{H}'(z) - E^{R}\Lambda(z) \text{ for } z \in R. \tag{4.9}$$

Hence

$$\Phi''(z) = (E^{L} - \bar{E}^{R})^{-1}[\text{H}'(z) - \bar{E}^{R}\Lambda(z)] \text{ for } z \in L, \tag{4.10}$$

$$\bar{\Phi}''(z) = (E^{R} - \bar{E}^{L})^{-1}[\text{H}'(z) - E^{R}\Lambda(z)] \text{ for } z \in R. \tag{4.11}$$

Now the heat flux is specified over both faces of the cut and hence, from (3.24), (3.25), (4.1), (4.4), (4.5), (4.10) and (4.11) it follows that

$$(E^{L} - \bar{E}^{R})^{-1}[\text{H}'^{+}(x_1) - \bar{E}^{R}\Lambda^{+}(x_1)] + (E^{R} - \bar{E}^{L})^{-1}[\text{H}'^{-}(x_1) - E^{R}\Lambda^{-}(x_1)] = -f(x_1) \text{ for } |x_1| < a. \tag{4.12}$$

$$(E^{R} - \bar{E}^{L})^{-1}[\text{H}'^{-}(x_1) - E^{R}\Lambda^{-}(x_1)] + (E^{L} - \bar{E}^{R})^{-1}[\text{H}'^{+}(x_1) - \bar{E}^{R}\Lambda^{+}(x_1)] + \Lambda^{-}(x_1) - \Lambda^{+}(x_1) = -f(x_1) \text{ for } |x_1| < a, \tag{4.13}$$

where $f(x_1)$ is the specified heat flux which is assumed to be the same on both faces of the crack. Subtraction of (4.13) from (4.12) yields

$$\Lambda^{+}(x_1) = \Lambda^{-}(x_1) \text{ for } |x_1| < a \tag{4.14}$$

and hence the function $\Lambda(z)$ is analytic in the whole plane. Furthermore since the temperature field must tend to zero as $|z| \to \infty$ it follows that $\Lambda(z)$ is identically zero. Hence (4.12) and (4.13) reduce to

$$(E^{L} - \bar{E}^{R})^{-1}\text{H}'^{+}(x_1) - (\bar{E}^{L} - E^{R})^{-1}\text{H}'^{-}(x_1) = -f(x_1) \text{ for } |x_1| < a, \tag{4.15}$$

or

$$\text{H}'^{+}(x_1) - \mu\text{H}'^{-}(x_1) = -(E^{L} - \bar{E}^{R})f(x_1) \text{ for } |x_1| < a, \tag{4.16}$$

where

$$\mu = (E^{L} - \bar{E}^{R})/(\bar{E}^{L} - E^{R}). \tag{4.17}$$

The solution to this Hilbert problem is

$$\mathrm{H}'(z) = \frac{-\chi(z)}{2\pi i} \int_{-a}^{a} \frac{(E^L - \bar{E}^R) f(x_1)\, dx_1}{\chi^+(x_1)(x_1 - z)}, \tag{4.18}$$

where

$$\chi(z) = (z-a)^{m-1}(z+a)^{-m} \tag{4.19}$$

$$m = \frac{1}{2\pi i} \log \mu \tag{4.20}$$

where we select the branch of $\chi(z)$ such that $z\chi(z) \to 1$ as $|z| \to \infty$ and choose the argument of μ to lie between 0 and 2π.

Having obtained the $\mathrm{H}'(z)$ from (4.18) the $\Phi''(z)$ may be obtained from (4.10), (4.11). Equations (4.4), (4.5) and (4.14) then provide $\bar{\Psi}''(z)$. The complete temperature and flux fields may then be evaluated from (3.24), (3.25) and (4.1).

5. THE DISPLACEMENT AND STRESS FIELDS

The displacement across $x_2 = 0$ for $|x_1| > a$ will be continuous if

$$B^L_{kj}\Omega_j^+(x_1) + \bar{B}^L_{kj}\bar{\Omega}_j^-(x_1) + C_k{}^L E^L \Phi^+(x_1) + \bar{C}_k{}^L \bar{E}^L \bar{\Phi}^-(x_1)$$
$$= B^R_{kj}\Theta_j^-(x_1) + \bar{B}^R_{kj}\bar{\Theta}^+(x_1) + C_k{}^R E^R \Psi^-(x_1) + \bar{C}_k{}^R \bar{E}^R \bar{\Psi}^+(x_1) \ \text{ for } |x_1| \geq a, \tag{5.1}$$

or

$$B^L_{kj}\Omega_j^+(x_1) - \bar{B}^R_{kj}\bar{\Theta}^+(x_1) + C_k{}^L E^L \Phi^+(x_1) - \bar{C}_k{}^R \bar{E}^R \bar{\Psi}^+(x_1)$$
$$= B^R_{kj}\Theta_j^-(x_1) - \bar{B}^L_{kj}\bar{\Omega}_j^-(x_1) + C_k{}^R E^R \Psi^-(x_1) - \bar{C}_k{}^L \bar{E}^L \bar{\Phi}^-(x_1) \ \text{ for } |x_1| \geq a, \tag{5.2}$$

where

$$B^L_{kj} = \sum_\alpha A^L_{k\alpha} M^L_{\alpha j}, \qquad B^R_{kj} = \sum_\alpha A^R_{k\alpha} M^R_{\alpha j}.$$

Hence if we put

$$B^L_{kj}\Omega_j(z) - \bar{B}^R_{kj}\bar{\Theta}_j(z) + C_k{}^L E^L \Phi(z) - \bar{C}_k{}^R \bar{E}^R \bar{\Psi}(z) = \Gamma_k(z) \ \text{ for } z \in L, \tag{5.3}$$

$$B^R_{kj}\Theta_j - \bar{B}^L_{kj}\bar{\Omega}_j(z) + C_k{}^R E^R \Psi(z) - \bar{C}_k{}^L \bar{E}^L \bar{\Phi}(z) = \Gamma_k(z) \ \text{ for } z \in R \tag{5.4}$$

where the functions $\Gamma_k(z)$ are analytic in the whole plane cut along $(-a, a)$ then eqn (5.2) is satisfied identically. Similarly the stress will be continuous across the bonded interface if

$$\Omega_i'(z) - \bar{\Theta}_i'(z) + (N^L_{i2} - \beta^L_{i2})E^L\Phi'(z) - (\bar{N}^R_{i2} - \beta^R_{i2})\bar{E}^R\bar{\Psi}'(z) = \Delta_i(x) \ \text{ for } z \in L, \tag{5.5}$$

$$\Theta_i'(z) - \bar{\Omega}_i'(z) + (N^R_{i2} - \beta^R_{i2})E^R\Psi'(z) - (\bar{N}^L_{i2} - \beta^L_{i2})\bar{E}^L\bar{\Phi}'(z) = \Delta_i(z) \ \text{ for } z \in R, \tag{5.6}$$

where the functions $\Delta_i(z)$ are analytic in the whole plane cut along $(-a, a)$.

Let

$$P_k{}^L(z) = C_k{}^L E^L \Phi(z) - \bar{C}_k{}^R \bar{E}^R \bar{\Psi}(z) \ \text{ for } z \in L, \tag{5.7}$$

$$P_k{}^R(z) = C_k{}^R E^R \Psi(z) - \bar{C}_k{}^L \bar{E}^L \bar{\Phi}(z) \ \text{ for } z \in R, \tag{5.8}$$

$$Q_i^L(z) = (N^L_{i2} - \beta^L_{i2})E^L\Phi'(z) - (\bar{N}^R_{i2} - \beta^R_{i2})\bar{E}^R\bar{\Psi}'(z) \ \text{ for } z \in L, \tag{5.9}$$

$$Q_i^R(z) = (N^R_{i2} - \beta^R_{i2})E^R\Psi'(z) - (\bar{N}^R_{i2} - \beta^L_{i2})\bar{E}^L\bar{\Phi}'(z) \ \text{ for } z \in R. \tag{5.10}$$

Hence from (5.5) and (5.6)

$$\bar{\Theta}_i'(z) = \Omega_i'(z) + Q_i^L(z) - \Delta_i(z) \ \text{ for } z \in L, \tag{5.11}$$

$$\Theta_i'(z) = \bar{\Omega}_i'(z) - Q_i^R(z) + \Delta_i(z) \text{ for } z \in R. \tag{5.12}$$

Hence differentiating (5.3) and (5.4) and substituting from (5.11) and (5.12)

$$(B_{kj}^L - \bar{B}_{kj}^R)\Omega_j'(z) - \bar{B}_{kj}^R Q_j^{L\prime}(z) + P_k^{L\prime}(z) = \Gamma_k'(z) - \bar{B}_{kj}^R \Delta_j(z) \text{ for } z \in L, \tag{5.13}$$

$$(B_{kj}^R - \bar{B}_{kj}^L)\bar{\Omega}_j'(z) - B_{kj}^R Q_j^R(z) + P_k^{R\prime}(z) = \Gamma_k'(z) - B_{kj}^R \Delta_j(z) \text{ for } z \in R. \tag{5.14}$$

Hence

$$\Omega_i'(z) = C_{ik}\{\Gamma_k'(z) - \bar{B}_{kj}^R \Delta_j(z)\} + C_{ik}\{\bar{B}_{kj}^R Q_j^{L\prime}(z) - P_k^{L\prime}(z)\} \text{ for } z \in L, \tag{5.15}$$

$$\bar{\Omega}_i'(z) = -\bar{C}_{ik}\{\Gamma_k'(z) - B_{kj}^R \Delta_j(z)\} - \bar{C}_{ik}\{B_{kj}^R Q_j^R(z) - P_k^{R\prime}(z)\} \text{ for } z \in R, \tag{5.16}$$

where

$$(B_{kj}^L - \bar{B}_{kj}^R)C_{jl} = \delta_{kl} \tag{5.17}$$

where δ_{kl} is the Kronecker delta. Use of (5.15) and (5.16) permits the boundary condition (2.3) to be written in the form

$$C_{ik}\{\Gamma_k'^+(x_1) - \bar{B}_{kj}^R \Delta^+(x_1)\} - \bar{C}_{ik}\{\Gamma_k'^-(x_1) - B_{kj}^R \Delta_j^-(x_1)\} = d_i(x_1) \text{ for } |x_1| < a \tag{5.18}$$

where

$$d_i(x_1) = -p_i(x_1) - C_{ik}\{\bar{B}_{kj}^R Q_j^{L+}(x_1) - P_k^{L\prime+}(x_1)\} + \bar{C}_{ik}\{B_{kj}^R Q_j^{R-}(x_1) - P_k^{R\prime-}(x_1)\}$$
$$- (N_{i2}^L - \beta_{i2}^L)E^L\Phi'^+(x_1) - (\bar{N}_{i2}^L - \beta_{i2}^L)\bar{E}^L\bar{\Phi}'^-(x_1). \tag{5.19}$$

Also use of (5.11), (5.12), (5.15), (5.16) and (3.34) permits the boundary condition (2.4) to be written in the form

$$C_{ik}\{\Gamma_k'^+(x_1) - \bar{B}_{kj}^R \Delta_j^+(x_1)\} - \bar{C}_{ik}\{\Gamma_k'^-(x_1) - B_{kj}^R \Delta_j^-(x_1)\} - \Delta_i^+(x_1) + \Delta_i^-(x_1) = g_i(x_1) \text{ for } |x_1| < a, \tag{5.20}$$

where

$$g_i(x_1) = -p_i(x_1) - C_{ik}\{\bar{B}_{kj}^R Q_j^{L+}(x_1) - P_k^{L\prime+}(x_1)\} + \bar{C}_{ik}\{B_{kj}^R Q_j^{R-}(x\;) - P_k^{R\prime-}(x_1)\} + Q_i^{R-}(x_1) - Q_i^{L+}(x_1)$$
$$- (N_{i2}^R - \beta_{i2}^R)E^R\Psi'^-(x_1) - (\bar{N}_{i2}^R - \beta_{i2}^R)\bar{E}^R\bar{\Psi}'^+(x_1). \tag{5.21}$$

Use of (5.9) and (5.10) in (5.21) shows that

$$g_i(x_1) = d_i(x_1) \tag{5.22}$$

and hence subtraction of (5.20) from (5.18) yields

$$\Delta_i^+(x_1) = \Delta_i^-(x_1) \tag{5.23}$$

and $\Delta(z)$ is analytic in the whole plane. Furthermore $\Delta(z) \to 0$ as $|z| \to \infty$ and hence $\Delta(z)$ is identically zero. Equations (5.19) and (5.20) thus reduce to

$$C_{ik}\Gamma_k'^+(x_1) - \bar{C}_{ik}\Gamma_k'^-(x_1) = g_i(x_1) \text{ for } |x_1| < a. \tag{5.24}$$

Multiplying by constants R_i which are yet to be determined and summing over i it follows that

$$R_i C_{ik}\Gamma_k'^+(x_1) - R_i\bar{C}_{ik}\Gamma_k'^-(x_1) = R_i g_i(x_1) \text{ for } |x_1| < a. \tag{5.25}$$

The R_i are chosen such that

$$R_i C_{ik} = S_k, \qquad R_i \bar{C}_{ik} = \lambda S_k \tag{5.26}$$

where the S_k and λ are yet to be determined. Elimination of the S_k provides

$$(\bar{C}_{ik} - \lambda C_{ik}) R_i = 0. \tag{5.27}$$

These equations have a non-trivial solution if

$$|\bar{C}_{ik} - \lambda C_{ik}| = 0 \tag{5.28}$$

which is a cubic in λ with roots λ_γ ($\gamma = 1, 2, 3$); the corresponding values of R_i and S_i obtained from (5.27) and (5.26) will be denoted by $R_{\gamma i}$ and $S_{\gamma i}$. Equation (5.25) may now be written

$$\{S_{\gamma k}\Gamma_k'^{+}(x_1)\} - \lambda_\gamma\{S_{\gamma k}\Gamma_k'^{-}(x_1)\} = R_{\gamma i} g_i(x_1) \text{ for } |x_1| < a. \tag{5.29}$$

The solution to this Hilbert problem is

$$S_{\gamma k}\Gamma_k'(z) = \frac{X_\gamma(z)}{2\pi i} \int_{-a}^{a} \frac{R_{\gamma i} g_i(x_1)\,dx_1}{X_\gamma^{+}(x_1)(x_1 - z)} \text{ for } \gamma = 1, 2, 3, \tag{5.30}$$

where

$$X_\gamma(z) = (z - a)^{n-1}(z + a)^{-n}, \tag{5.31}$$

$$n = \frac{1}{2\pi i} \log \lambda_\gamma,$$

where we select the branch of $X_\gamma(z)$ such that $zX_\gamma(z) \to 1$ as $|z| \to \infty$ and choose the argument of λ_γ to lie between 0 and 2π. Equation (5.30) provides

$$\Gamma_k'(z) = \sum_\gamma \left\{\frac{T_{k\gamma} X_\gamma(z)}{2\pi i} \int_{-a}^{a} \frac{N_{\gamma i} g_i(x_1)\,dx_1}{X_\gamma^{+}(x_1)(x_1 - z)}\right\}, \tag{5.32}$$

where

$$S_{\alpha k} T_{k\beta} = \delta_{\alpha\beta}.$$

Having obtained the $\Gamma_k'(z)$ from (5.32) it is possible to substitute back into (5.13) and (5.14) to obtain $\Omega_j'(z)$ and then (5.3) and (5.4) then yield $\Theta_j(z)$. The displacement and stress throughout the bonded material may then be obtained from (3.31) to (3.34).

6. STRESS NEAR THE CRACK TIP

In this section the nature of the stress singularities near the crack tip is considered. Suppose the flux over the crack faces is constant and equal to f_0. Then eqn (4.18) may be integrated to yield

$$\text{H}'(z) = \frac{-f_0(E^L - \bar{E}^R)}{(1 - e^{2\pi m i})}\{1 - [(2m - 1)a + z]\chi(z)\}. \tag{6.1}$$

Hence

$$\text{H}(z) = \frac{-f_0(E^L - \bar{E}^R)}{(1 - e^{2\pi m i})}\{z - (z - a)^m (z + a)^{-m+1}\}. \tag{6.2}$$

where m is given by (4.20).

From (5.21), (5.7) to (5.10), (4.8) to (4.11), (4.4) and (4.5) it follows that

$$g_i(x_1) = -p_i(x_1) + h_i \text{H}^{+}(x_1) + k_i \text{H}^{-}(x_1), \tag{6.3}$$

where

$$(E^L - \bar{E}^R)h_i = -C_{ik}[\bar{B}^R_{kj}\{(N^L_{j2} - \beta^L_{j2})E^L - (\bar{N}^R_{j2} - \beta^R_{j2})\bar{E}^R\} - (C_k{}^L E^L - \bar{C}_k{}^R \bar{E}^R)] - (N^L_{i2} - \beta^L_{i2})E^L, \tag{6.4}$$

$$(E^R - \bar{E}^L)k_i = \{\bar{C}_{ik}[B^R_{kj}\{(N^R_{j2} - \beta^R_{j2})E^R - (\bar{N}^L_{j2} - \beta_{j2})\bar{E}^L\} - (C_k{}^R E^R - \bar{C}_k{}^L \bar{E}^L)]\} - (\bar{N}^L_{i2} - \beta^L_{i2})\bar{E}^L. \tag{6.5}$$

Use of (6.2) in (6.3) provides

$$g_i(x_1) = -p_i(x_1) - \frac{f_0(E^L - \bar{E}^R)(h_i + k_i)x_1}{(1 - e^{2\pi m i})} + \frac{f_0(E^L - \bar{E}^R)}{(1 - e^{2\pi m i})}[h_i + k_i\, e^{(-m+1)2\pi i}]\chi_1{}^+(x_1), \tag{6.6}$$

where

$$\chi_1(z) = (z - a)^m (z + a)^{-m+1}. \tag{6.7}$$

If the applied tractions are constant then $p_i(x_1) = t_i$ (constant) and (6.6) and (6.7) may be substituted into (5.23) which, upon integration, yields

$$\begin{aligned}\Gamma_k'(z) = \sum_\gamma T_{k\gamma} X_\gamma(z) \Bigg[&\frac{N_{\gamma i} t_i}{(1 - e^{2\pi n i})}\left\{\frac{1}{X_\gamma(z)} - [(2n-1)a + z]\right\} \\ &- \frac{f_0(h_i + k_i)(E^L - \bar{E}^R)}{(1 - e^{2\pi m i})(1 - e^{2\pi n i})}\left\{\frac{z}{X_\gamma(z)} - [(z_0 + na)(z_0 + n - 1) + n(n-1)a^2]\right\} \\ &+ \frac{f_0(E^L - \bar{E}^R)}{(1 - e^{2\pi m i})}\frac{(h_i + k_i\, e^{2\pi i(1-m)})}{2\pi i}\int_{-a}^{a}\frac{\chi_1{}^+(x_1)\,dx_1}{X_\gamma{}^+(x_1)(x_1 - z)}\Bigg],\end{aligned} \tag{6.8}$$

where

$$\int_{-a}^{a}\frac{\chi^+(x_1)\,dx_1}{X_\gamma{}^+(x_1)(x_1 - z)} = \begin{cases} \dfrac{2\pi i}{(1 - e^{2\pi i(n-m+1)})}\left[\dfrac{\chi_1(z)}{X_\gamma(z)} - \{2az(n-m) + z^2 + 2a^2(n-m)^2 - a^2\}\right] & \text{if } n \neq m \\ 2az + (z^2 - a^2)\log\left(\dfrac{z-a}{z+a}\right) & \text{if } n = m. \end{cases} \tag{6.9}$$

The stress in the bonded material is given by linear combinations of the analytic functions $\Gamma_k'(z)$ and hence (6.8) and (6.9) provide the relevant information about the nature of the stress singularities at the crack tip. Note first that the type of stress singularity at the crack tip is given by the functions $X_\gamma(z)$ for $\gamma = 1, 2, 3$ and these functions are the same regardless of the presence or magnitude of the applied heat flux. Hence the comments and calculations made about the stress singularities for some particular anisotropic elastic materials in Clements[6] are also pertinant to the thermoelastic case.

In the case when the two half-spaces consist of the same anisotropic material it follows from (3.23) and (4.17) that $\mu = -1$ so that $m = (1/2)$. Also, in this case, it follows from (5.17) that C_{jl} has zero real part so that $C_{jl} = -\bar{C}_{jl}$. Hence when the half-spaces are the same (6.4) and (6.5) in conjunction with (5.17) provide

$$\begin{aligned} h_i + k_i\, e^{(-m+1)2\pi i} &= h_i - k_i \\ &= 0. \end{aligned}$$

Thus, in this case, the last terms in (6.6) and (6.8) are zero and the stress field induced by the constant heat flux in the absence of applied tractions may be obtained by taking the $g_i(x_1)$ in (6.6) in the form

$$g_i(x_1) = -f_0 h_i (E^L - \bar{E}^R) x_1.$$

If the half-spaces are different then the stress field induced by the constant heat flux in the absence of applied tractions is given by putting $t_i = 0$ in (6.8) and then substituting the resulting $\Gamma_k'(z)$ in the relevant expressions in order to obtain the stresses.

7. SUMMARY

Complex variable techniques have been employed to solve the problem of determining the flux and stress fields round a cut along the joint between two dissimilar anisotropic half-spaces. A closed form solution has been obtained for the case of constant flux and tractions over the crack faces. The analysis indicates that the nature of singularities in the stress field at the crack tips is unaltered by the presence of an applied heat flux over the crack faces. The introduction of the heat flux simply modifies the coefficients of the singularities.

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